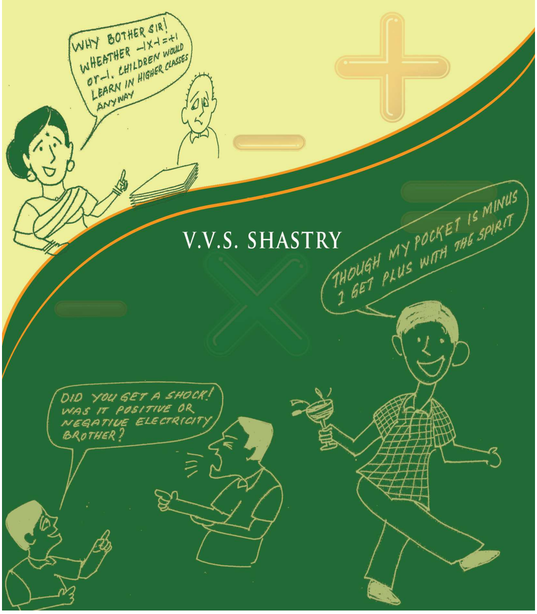
MANY APPROACHES TO

$-1 \times -1 = +1$

V.V.S. SHASTRY



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Contents

Preface

I have been travelling throughout the country as a MATH communicator. After an interesting lecture, people would come to me to ask

“why is -1 X-1=+1?”.

I heard that few people would give a wrong answer such as – it is an axiom. Hence I began to search possible proofs for this vexed problem. Modern mathematics proves this through Group theory. But for secondary school level children, this higher mathematics is incomprehensible. Hence even though these approaches may not satisfy a philosopher-mathematician, they will suffice a curious secondary school student. Therefore this book is expected to satisfy lots of school children.

Foreword

On mathematical operations

Addition and deletions are taught in schools without resorting to much explanation. But when teaching multiplication which is multiple addition, some effort is required.

Let us see this multiplication

$$+a \cdot +b = +ab$$

There are three elements to be observed here:

1. a → To be multiplied (a value)

2. + → What is to be multiplied has a positive value

3. b → To multiply with this number

4. + → You have to carry out multiplication under the same sign as multiplicand. Imagine the same operation is done with – b instead of + b .

   Then b multiplayer – How many times multiplicand has to be added repeatedly

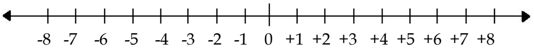
– Carry out multiplication by the inverse sign of the multiplicand.
Observe here that –1 is the inverse of +1

We have to keep these things in mind to understand following pages

Where is the confusion?

Mathematics teaching gets started with the introduction of positive numbers. This is usually done using Number line.

When the negative numbers get introduced the same number line is extended. The number line starts from 0 and is stretched to right side of the page. The same line is extended now to the left side of the page.



In this arrangement negative numbers run opposite to the direction of positive numbers on the number line. And each negative number finds a place exactly at a distance opposite to the corresponding positive numbers on right side of the number line, like an image reflected in a mirror.

This sticks to the students mind. When the Left side on number line is a mirror image of the Right side why can't the Mathematical operation like $+1 \cdot +1 = +1$ Shouldn't find its counterpart as $-1 \cdot -1 = -1$? This is root cause of all discomfort a student finds to accept $-1 \cdot -1 = +1$

When number line is written in this way, does it show all aspects of number we are supposed to represent? For example

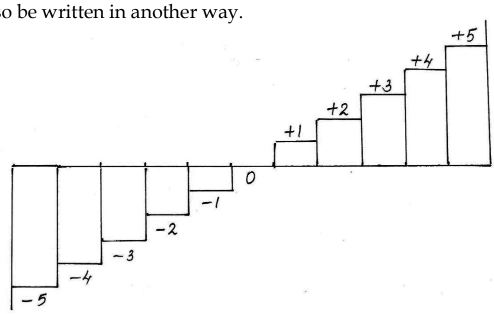
a) On this number line one cannot find out at first sight, which numbers are bigger and which are smaller.

b) When you move to the right, from any point on the number line the value of numbers goes on increasing

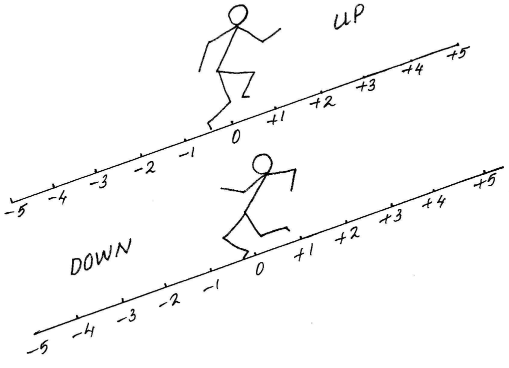
c) When you move to the left, from any point on the same number line, the value of numbers decreases.

Then why not we write the number line as below?

Then (a) (b) & (c) are more obvious. Number Line can also be written in another way.



In this example also (a) (b) & (c) are more obvious. One aspect is crucial. Both differ from earlier number line in that they do not show positive and negative number as mirror images. But numbers spread out in opposite directions with zero at the centre. In other words for any positive number there is an inverse number and it is glaring.



Know this

Negative number did not come in to circulation until the concept ZERO was understood.

We can ask what is so difficult about Zero to be understood?

Because Zero = nothing, cifer etc. are not Zero in the number system. But it is a number with its value zero

For example "Ramesh has a Zero bank balance"

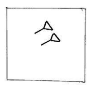
This has two meanings:-

(a) Ramesh has no Bank A/c. So his savings are nil. Absence of an A/c implies balance is Zero

(b) Ramesh has a Bank A/c. The balance in it is Zero. So Zero in both cases are different.

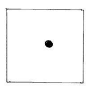
Only three civilizations used Zero in human history

- Babylon –the earliest record of zero is 1850 BC

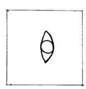
  Their symbol for Zero

- Indian – The earliest document is dated 6thAD

  Their symbol for Zero

- Maya (America) used Zero. It was in 10^{th} AD. Their symbol for Zero

Note this:

Take this number. one thousand nine hundred and nine. Between 1 & 9 there is nothing – nil. But between 9 & 9 there is Zero – 0. That means in 'tens' place there is no number of any value. Hence it is represented by '0'.

Symblos used in mathematical operations

+ and − Signs were in use even before printing was invented. These signs were used on wine barrels to show that they are full (-) and not full (+).

+ *is from Latin word et = meaning 'and'*

+& − signs were first printed in 1490 in a book titled Mercantile Arithmetic – by Johannes Whitman

It was David Recarde who used them in 1557 in modern sense.

· sign for multiplication was used by Willian Oughtred in 1631 in a book "key to Mathematics "

G. W. Leibnitz wrote a letter dated 29.7.1698 to John Bernuoulli. In that letter he wrote 'I do not want to use X symbol for multiplication becasue it is similar to X used in Algebra and an unknown. So I write a dot ZC. LM to show multiplication. To write division let us use two dots **(:)**

Approach – 1

Finding a pattern in multiplication. Observe the following

a) $4 . 3 = 12$

b) $(4 – 1) \times 3$
$3 \cdot 3 = 9$

c) $(3 – 1) = 3$
$2 \cdot 3 = 6$

d) $(2 – 1) \cdot 3$
$1 \cdot 3 = 3$

e) $(1 – 1) \cdot 3$
$0 \cdot 3 = 0$

f) $(0 – 1) \cdot 3$
$–1 \cdot 3 = -3$

g) $3 \cdot (–1) = -3$

h) $(3 – 1) \cdot (-1)$
$2 \cdot (–1) = -2$

i) $(2 – 1) \cdot (–1)$
$1 \cdot (–1) = –1$

j) $(1 – 1) \cdot (-1)$
$0 \cdot (–1) = 0$

k) $(0 – 1) \cdot (–1)$ (–
1) $\cdot (–1) = + 1$

- From (a) to (k) multiplication of two numbers are Shown.

- At every stage multiplicand gets reduced by 1. So 4 has reduced to –1 in 6 stages and 3 also reduced to –1 in 4 stages

- When 1 gets subtracted from the multiplicand, what effect it has on the multiple?

u The effect is to add – 3 to the previous multiple. Ex : 12 – 3 = 9 up to the stage (f) and the effect is to add –1 to the previous multiple from step (h) – (k) Ex: –2 + 1 = –1.

Approach – 2

Data to be known:

'a' and ' b ' represent two numbers,

Zero can be represented as
(a number + its additive inverse)
Ex: $x + (-x) = 0$

\endash Algebraic operation – addition by 0 is nothing
but addition by number and its inverse,

Taking a common factor, etc
Now consider
-a × –b = –a × – b + 0

= (-a × –b) + (ab – ab) (Replacing '0' by (ab-ab))

=+ab +(-a × b)+(-a × -b) (rearranging the terms)
= +ab – a (b – b) (since –a is common)
=+ab-a(0)
=+ab
Therefore *-a* × *-b* = +*ab*

Take a = 1, b = 1 you get ***(-1) × (-1) = +1***

Approach – 3

$-1 \times -1 = +1$

Multiplication method – using Zero

Data to be known

a) Zero can be written as $+1 - 1 = 0$

b) $-1 \times +1 = -1$

Now consider $-1 \times -1 = 0 + (-1 \times -)$

$=(+1 - 1) + (-1 \times -1)$

$-1 \times -1 = +1\ (-1 \times +1) + (-1 \times -1)$ take -1 as common factor

d) $+1 + 1\ (-1)\ (+1 -1)$

e) $+1 + 0 = +1$

One may ask why should we take $0 = -1 + 1$

Can't it be $-2 + 2 = 0$

Let us see what happens when we take $0 = +2 - 2$

$-1 \times -1 = 0 + (-1 \times -1)$

$=+2 - 2 + (-1 \times -1)$

$=+2 - 1 - 1 + (-1 \times -1)$

$=+2 - 1 - 1\ (+1 -1) = 2 - 1 = +1$

Approach – 4

–1 × –1 = +1

Considering electrical charges

Prior knowledge required

a) Let us give a value 1=one charge

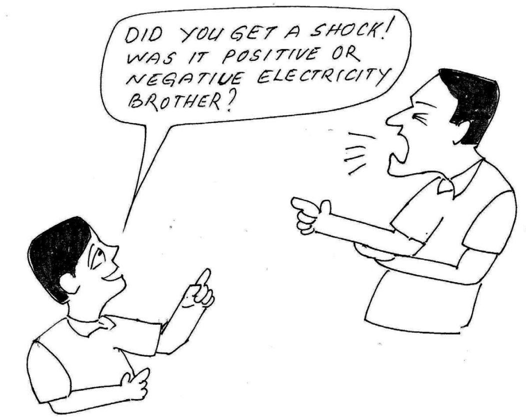
b) Positive charges are positive numbers

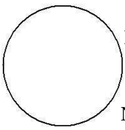
c) Negative charges are negative numbers

d) One positive charge + one negative charge = 0

That is + 1 – 1 = 0

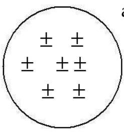
Consider that electrical charges exist in an electrical field





Here is a area. There are no electric charges here.

So can we say "0 "Electric charge is here? No, we need not use '0' here, as it is not a field at all



Here is an electric field. The total electric charge in the field is Zero. Because it has as many positive charges as negative charges. Total effect is Zero charge.

To start the experiment we have to assume the following rules.

a) To put electrical charges in to the 'filed' means addition.

b) To take out electric charges out of 'filed' means subtraction.

Therefore Addition = positive activity
Subtraction = negative activity

Now consider this:

In a field where total effect of charges is Zero charge, takeout two negative charges two times out of the field. Then what do you get –

$$-2 \times -2 = +4$$

(Two negative charges) × (taking out two times) = what remains are 4 positive charges.

Approach – 5

–1 × –1 = +1

Through the number line

Prior knowledge required:

a) To measure positive/negative numbers on the number line

b) When you proceed from left to right it is as good as adding numbers

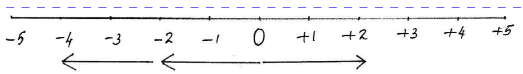
c) When you proceed from right to left it is a good as subtracting numbers.

d) Moving along the direction of measured number on the line – positive activity.

e) Moving opposite the direction of measured numbers on the line – negative activity.

How to do the activity to get +3 × –2 =- 6 ?

To get +3 Measure 3 lengths to the right of Zero. Mark this as one unit.

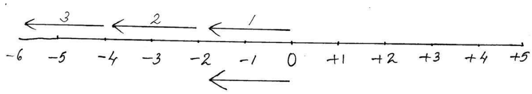


Comment [uS1]: Delete current fig and insert S1

To get –2 Move this unit (of 3 lengths) in the direction opposite to the direction of first measurement, to a distance of two lengths, make a mark where it ends.

It is then $+ 3 \times -2 = - 6$

To comprehend $-2 \text{ X} + 3 = -6$



To get - 2 start from Zero, count two to the left. Make this as one unit.

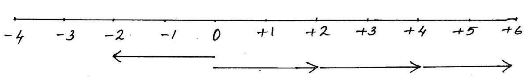
To multiply by + 3 → Move this 'unit' in the same direction as you counted -2 – that is to the Left

It is then $-2 \cdot +3 = -6$
To comprehend $- 2 \cdot - 3 = + 6$

To get – 2 → Start from zero count two lengths to the left. Make this one 'unit'.

To multiply by –3 → move the 'unit' in opposite to the direction, in which –2 was measured. that is to the right from Zero.

Then $-2 \text{ X} -3 = + 6$



Approach – 6

$-1 \cdot -1 = +1$

Thro Activity:

Materials required

i) 12 card board tokens of equal size. Write + 1 on 6 tokens on one side.

j) Write – 1 on the other 6 tokens on one side.

k) one small card board box

l) Putting tokens inside the box is +ve action.
Putting tokens out of the box is –ve action

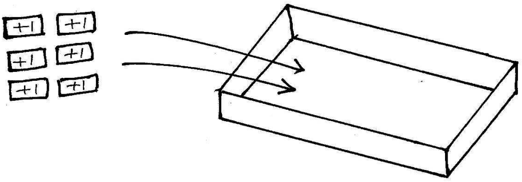
Things to be known before activity:-

a) Each token represents + 1 or – 1 as written on it.

b) Putting token inside the box is +ve activity. Taking the token out of the Box is –ve activity.

c) When + token and – 1 tokens are together in equal number, the total value inside the tray will be considered as '0'

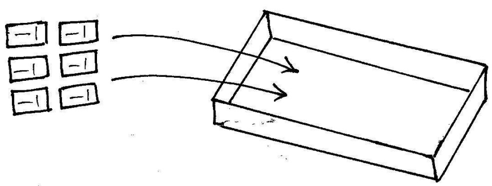
1) Activity to get **+3** X **+2 = +6**



Put 3 tokens with + 1 written on them inside the box two times. And count. That means +3 x +2 = +6

2) Activity to get –3* x *+2 = –6

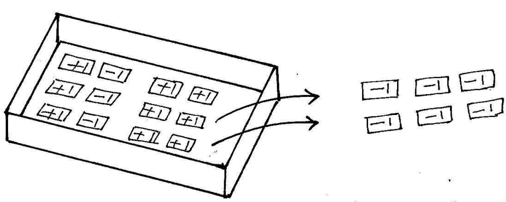
Put three tokens which have – 1 on it, in to the box and count. That means –3 x +2 = –6



3) Activity to get -3* X *-2 = + 6

There is nothing inside the Box. That means no tokens are inside the Box. Then to ask the value of Tokens inside the box is absurd. But you put as many +1 and as many -1 (pairs) into the box. Then you can ask the value of total tokens present inside the box. It is equal to 'Zero'.

Now when total value of tokens is zero, take away three -1 Tokens, Two times out of the box and then count Total value of tokens present inside the box. You have +6 tokens giving value that means –3 · –2 = +6



Approach – 7

–1 X –1 = +1

On number line – addition/subtraction.

Prior knowledge required:

a) Measuring and marking positive and negative numbers on number line

b) To add means, measuring in terms of unit lengths from Zero. To subtract means measuring unit lengths in the opposite direction.

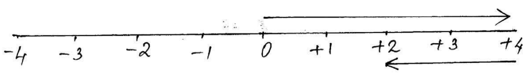
*1) **To add positive numbers:*** **+ 2 + 2 = + 4**

Measure two lengths from '0 'to the right of the number line. This is unit length of '2'. Measure two lengths with this unit. You can read 4 the line.

That is **+ 2 + 2 = + 4**

2) To subtract positive numbers : +4 – (+2) = +2

Measure 4 lengths from '0' to the right. Then measure 2 lengths in opposite direction.



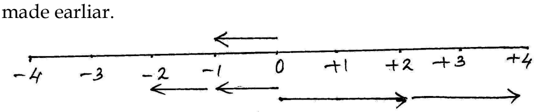
That is +4 – (+2) = +2

3) Adding negative numbers: –2 + (–2) = –4

Consider the number line:

Measure two lengths from '0' to the left of the Number line, and mark it as unit.

Measure two units in the same direction from the mark made earliar.



The number reads – 4

That means –2 + (–2) = -4

4) Subtracting negative numbers: -2-(-2)=0

Measure two units from '0' to the left, on the number line, and make a mark.

Measure two units starting from the same marked point in opposite direction.

Comment [uS2]: Replace current fig with S2

That means $-2 - (-2) = 0$

What has happened here?
Is it not the same as $-1 \cdot -1 = +1$?
when we do $-(-2) = +2$

Approach – 8

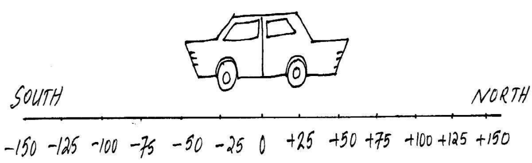
Through the analogy of a speeding car

Point to remember before the activity:

5. Speeding up towards the north is +ve
6. Speeding up towards the south is –ve
7. Covering the distance towards north +ve
8. Covering the distance towards south –Ve

   Time to be taken +ve

   Time taken already –ve



1) *You are now at '0' in a car. You travel towards north at 50 km/hr. After travelling for 3hrs, where will you be?*

+50 **X** **+3** = **150**

(speed towards north) (time required) (you will cover distance after '0')

2) Now having travelled from south, you have arrived at '0'. The speed of the car was 50 km/hr. Where were you 3 hrs ago?

+50	**X**	**-3**	**=**	**-150**
(speed towards north)		(time passed)		(you were 150 km behind '0')

3) You are at '0'. Now you are travelling towards south at 50km/hr. Where will you be after 3 hours from now?

-50	**X**	**+3**	**=**	**-150**
(speed towards south)		(time required)		(After 3 hours you will be at the distance to the south'0')

4) You have started travelling from north towards South and arrived at '0'

Your Speed was 50km/hr.

Where were you 3 hrs ago?

–50	**X**	**–3**	**=**	**+150**
(Speed towards South)		(Time passed)		(you were at this from '0' towards north)

Approach – 9

Profit/loss method

Prior knowledge required:

a) Numbers represent money in rupees
b) Profit means +ve Loss means –ve
c) Prior period –ve, future period +ve.

Consider a businessman who is undergoing a loss of Rs.100 per day. Therefore

a) The loss to the Businessman for 3 days after (from today)

Rs. –100	·	+3	=	–300
(Loss)		(1 day before)		

b) Three days before (from today)the buisinessman had Rs 300/more than what he is having today. That means

Rs. –100	·	–3	=	+300
(Loss)		(days before)		

Approach – 10

$-1 \cdot -1 = +1$

ORIGAMI – paper folding

method. Your require:-

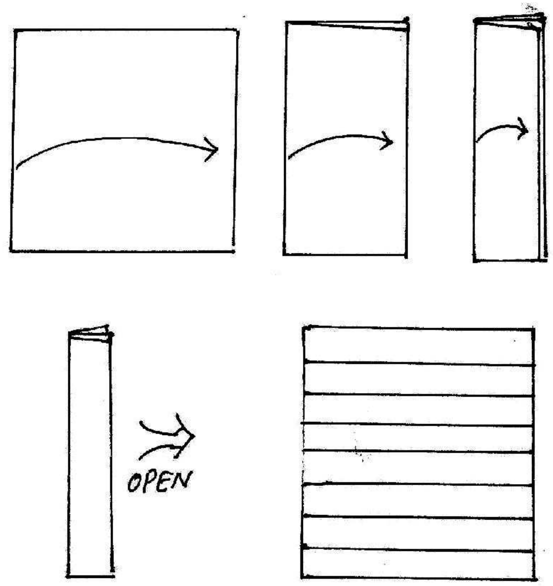
e) 15 cm X 15 cm. White paper

f) Knowledge about similar triangles.

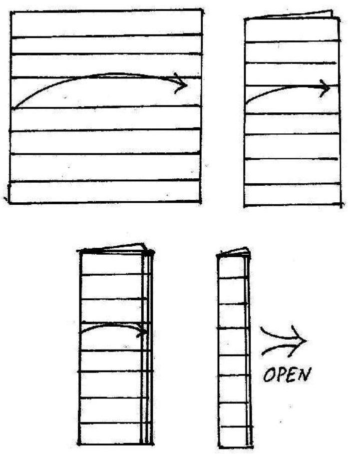
g) Knowledge about gr aphs

Fold a square paper as shown below.

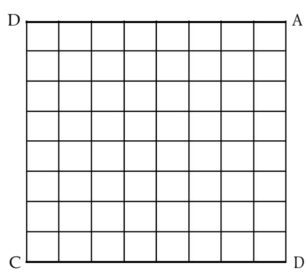
Comment [uS3]: Replace fig with S3



Now turn the folded paper horizontally and repeat folding



You get a square with 64 small squares in it. Use this tessellated square as a graph sheet

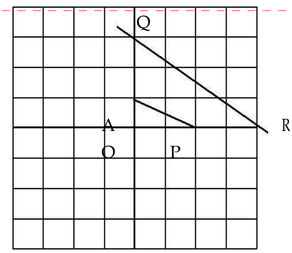


In any graph we have to define a unit to measure.

Therefore, let OA = 1 unit

Mark OP=2 and OQ on graph, Join P and A.

From Q draw a line parallel to AP which joins at R



Comment [S4]: Replace fig with S4

Now Δ OAP is similar to Δ OQR

Therefore $\frac{OA}{OQ} = \frac{OP}{OR} = \frac{AP}{QR}$

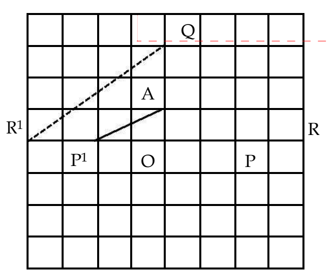
Comment [S5]: Adjust "=" in equations

We require OQ · OP = OA, OR = (1) · OR

That means +2 · +2 = 4

Measure OR which will be equal to 4 units.

To prove +2 · +2 - =4



Comment [S6]: Replace fig with S5

We already have OQ = +2,

Now measure OP’ = –2,
From the point Q draw a parallel to AP

NowOAP^1 is similar = ORQ

Therefore $\frac{OA}{OQ} = \frac{OP'}{OR'} = \frac{AP'}{QR'}$

Comment [S7]: Adjust “=” in equations

We require

2 · +2 + OP · OQ = OA · OR Measure OR = 4 units
=(+1) · (–4)
=–4

Therefore -2 · +2 = –4

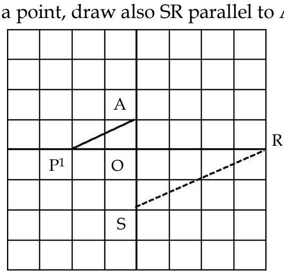
III To prove -2 · -2 = 4

We already have OP = +2,

Measure OS = –2 units.

AP’ Is already joined

With S as a point, draw also SR parallel to AP



Now we have Δ APO similar to Δ OSR

Therefore $\frac{OP}{OR} = \frac{OA}{OX}$

$-2 \cdot -2 = OP., OS = OA . OR$ Measure OR = 4 units

$= (+1)\ (+4) \ = 4$

$-2 \cdot -2 = +4$

Comment [S8]: Adjust "=" in equations

Approach – 11

You require–

a) A small Tray

b) Six paper pieces of equal size, .

c) Six paper pieces can represent numbers. Each paper has +1 value

d) The pieces of paper found inside the tray have positive value.

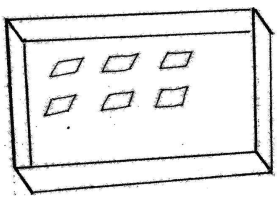
e) The pieces of paper found outside the tray have negative value.

f) Activity which involves displacement of paper pieces are negative.

g) Activity which takes place without displacement of paper pieces is positive.

Activity I

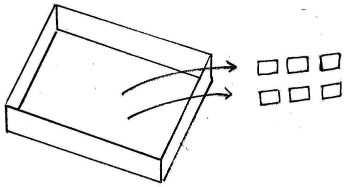
$-1 \cdot -1 = +1$



Six pieces of paper is kept inside the tray. Arrange these pieces of paper into two rows of three pieces each. This can be represented as

+3	·	+2	= +6
(Pieces picked inside the tray)		(arranged twice at the someplace)	(The Number of pieces found inside the tray after activity)

Activity II



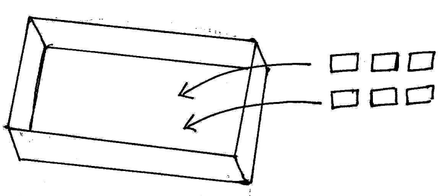
Pick three pieces of paper and arrange them in a row. Do it Two times Outside the Tray

+3	· –2	= –6
(Pieces picked inside the tray)	(No. Of times displaced)	(No. of pieces found after activity. Outside the tray)

Activity III

Pick three pieces of paper found outside the tray and put them in a row inside the Tray – Two times.

–3	· –2	= +6
(Pieces picked outside the tray)	(No. of times displaced)	(No. of pieces found inside the tray ofter)



ENTERTAINMENT

Few jokes are circulating about multiplication of negative numbers.

Doing Good work	+	Positive
Not doing Good work	–	Negative
Good	+	Positive
Evil	–	Negative

Always doing good work : $+1 \cdot +1 = +1$ always good
Never doing good work
$+1 \cdot -1 = -1$ always bad
Always doing bad work ______________bad
$-1 \cdot +1 = -1$
Never doing bad work _______________good
$-1 \cdot -1 = +1$